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Soliton-like magnon localization and the two-magnon bound state in a Heisenberg ferromagnetic chain

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Abstract. An inhomogeneous Heisenberg spin Hamiltonian with single-ion anisotropy is used to investigate the non-linear excitations in a ferromagnetic chain. By means of the Holstein–Primakoff transformation and Glauber's coherent-state representation, the equation of motion for the annihilation operator $a(j)$ is reduced to a non-linear Schrödinger-like equation with variable coefficients. Its non-linear modified terms are strongly restricted by the relation between the continuum approximation ($\eta = a/\lambda_0$, where η is the 'degree' of the long wavelength, a is the lattice constant and λ_0 the characteristic wavelength of the excitation) and the semiclassical approximation ($\varepsilon = 1/\sqrt{S}$, where ε is the degree of truncation of the operator expansion and S is the spin length). When assuming that $\eta = \mathrm{O}(\varepsilon)$ and after retaining the terms to $\mathrm{O}(\varepsilon^4)$, the motion of the coherent amplitude for the homogeneous case satisfies the non-linear Schrödinger equation. The single-soliton and two-soliton bound-state solutions are given. The results show that the magnon localization and two-magnon bound state are possible in the chain. Other relations between η and ε ($\eta = \mathrm{O}(\varepsilon^{1/2})$, $\eta = \mathrm{O}(\varepsilon^{3/2})$ and $\eta = \mathrm{O}(\varepsilon^2)$) are also discussed.

1. Introduction

The interaction of spin waves was first investigated by Bethe [1], Dyson [2] and Wortis [3]. It was shown that, for all spins and dimensionalities, bound states exist and, for a positive exchange constant and sufficiently large longitudinal anisotropy, there are bound states of two spin waves with energies below all continuum energies [1–3]. Silberglit and Torrance [4] pointed out that the system may create an additional bound state when the Hamiltonian contains a single-site anisotropy. By means of the method of equation of motion for Green functions, many physicists have studied the problem of the bound state of spin waves in recent years [5–10]. By taking the exchange constants $J(j, j')$ as random parameters, Salzberg *et al* [11] investigated the localization of spin waves in a disordered magnetic system by using the cluster Bethe lattice approach. They showed the existence of localized spin-wave states due to both isolated magnetic clusters and non-propagating magnon modes [11].

On the other hand, the non-linear excitations such as solitary waves or solitons in the Heisenberg spin chain have attracted considerable attention in recent years. Theoretically, there are several methods to study these non-linear excitations in one-dimensional magnets. In the classical method [12, 13], the general soliton solutions are obtained for a continuum version of the classical Heisenberg chain. The gauge equivalence between the Heisenberg ferromagnet and the non-linear Schrödinger system has also been shown [14]. For a quantum spin system, the bosonic representation of the spin operators turns out to be a very suitable method for studying soliton excitations since quantum corrections can be included in a systematic way. In the spin-coherent-state representation [15], one can work directly with spin operators, make no approximations to a Hamiltonian and obtain an exact non-linear equation of motion for the spin-coherent amplitude [16]. The other coherent-state treatments use a severely truncated operator expansion for $S^{\pm}$ [17–21] or an approximate Hamiltonian which is biquadratic in boson operators [22]. Working in Glauber's coherent-state representation and making the semiclassical approximation and the long-wavelength approximation, one then finds solitary-wave profiles of the system, which is the so-called semiclassical treatment.

The consistency and validity of the semiclassical treatment which has been widely used in the study of non-linear excitations in magnetic systems were re-examined in our recent work [23, 24]. It is argued that the modified terms of the equation of motion are strongly restricted by the relation between the semiclassical approximation and the long-wavelength approximation. In this paper we extend our approach to investigate the non-linear excitations in an inhomogeneous Heisenberg ferromagnetic chain with single-ion anisotropy. The paper is organized as follows. In section 2, we write the model Hamiltonian in dimensionless form and introduce the Holstein–Primakoff (HP) transformation and Glauber's coherent-state representation. The equation of motion is obtained in the semiclassical approximation and in the long-wavelength approximation. Single-soliton and two-soliton solutions are given in section 3. In section 4, we discuss several relations between η and ε. Section 5 is a summary and some comments.

2. The model Hamiltonian and the equation of motion

The system under consideration is described by the Hamiltonian

$$H - H_0 = -\sum_j J(j)[S(j)\cdot S(j+1) - S^2\hbar^2] - \sum_j D(j)[(S^z(j))^2 - S^2\hbar^2] - g\mu_B\sum_j f(j)[S^z(j) - S\hbar] \qquad (1)$$

where $J(j)$ are the exchange interaction parameters (assumed to be positive), g is the g-factor, μ_B is the Bohr magneton, $f(j)$ is the intensity of an external magnetic field applied in the z direction and $D(j)$ are the single-ion anisotropy constants. Here we have extended the Hamiltonian used in [4] to an inhomogeneous case where J, D and f are dependent on site j.

Introducing the definitions $\tilde{S}(j) = S(j)/\hbar$, $\tilde{J}(j) = J(j)/J_0$, $\tilde{D}(j) = D(j)/J_0$ and $\tilde{f}(j) = g\mu_B f(j)/J_0 S\hbar$ (J_0 is a typical exchange constant of the system), we can write the Hamiltonian (1) in the dimensionless form

$$\tilde{H} = -\sum_j \tilde{J}(j)\frac{[\tilde{S}^+(j)\tilde{S}^-(j+1) + \tilde{S}^-(j)\tilde{S}^+(j+1)]/2 + \tilde{S}^z(j)\tilde{S}^z(j+1) - S^2}{S^2} - \sum_j \tilde{D}(j)\frac{\tilde{S}^z(j)\tilde{S}^z(j) - S^2}{S^2} - \sum_j \tilde{f}(j)\frac{\tilde{S}^z(j) - S}{S} \qquad (2)$$

where $\tilde{H} = (H - H_0)/J_0 S^2\hbar^2$ and $\tilde{S}^{\pm}(j) = \tilde{S}^x(j) \pm \mathrm{i}\tilde{S}^y(j)$ with $\tilde{S}(j)\tilde{S}(j) = S(S+1)$. At low temperatures $(a^+(j)a(j) \ll 2S)$, the HP [25] transformation for the spin operators

$$\tilde{S}^+(j) = [2S - a^+(j)a(j)]^{1/2}a(j) \tag{3}$$

$$\tilde{S}^-(j) = a^+(j)[2S - a^+(j)a(j)]^{1/2} \tag{4}$$

$$\tilde{S}^z(j) = S - a^+(j)a(j) \tag{5}$$

can be expanded into the power series of $\varepsilon = 1/\sqrt{S}$ in the semiclassical approximation (i.e. $\lim_{\substack{S\to\infty \\ \hbar\to 0}} S\hbar = S_c$):

$$\tilde{S}^+(j)/S = \sqrt{2}\,[\varepsilon a(j) - \varepsilon^3 a^+(j)a(j)a(j)/4 - \varepsilon^5 a^+(j)a(j)a^+(j)a(j)a(j)/32 + \mathrm{O}(\varepsilon^7)] \tag{6}$$

$$\tilde{S}^-(j)/S = \sqrt{2}\,[\varepsilon a^+(j) - \varepsilon^3 a^+(j)a^+(j)a(j)/4 - \varepsilon^5 a^+(j)a^+(j)a(j)a^+(j)a(j)/32 + \mathrm{O}(\varepsilon^7)] \tag{7}$$

$$\tilde{S}^z(j)/S = 1 - \varepsilon^2 a^+(j)a(j) \tag{8}$$

where $a(j)$ and $a^+(j)$ are the Bose operators. After substituting (6)–(8) into (2) and then taking the Heisenberg equation of motion for $a(j)$, we obtain

$$\begin{aligned}
\mathrm{i}\tilde{\omega}_0\,\partial a(j)/\partial\tilde{t} = {} & \varepsilon^2\{[\tilde{f}(j) + 2\tilde{D}(j) + 2\tilde{J}(j)]a(j) - \tilde{J}(j-1)a(j-1) - \tilde{J}(j)a(j+1)\} \\
& + \varepsilon^4\{\tilde{J}(j-1)a^+(j)a(j)a(j-1)/2 + \tilde{J}(j)a^+(j)a(j)a(j+1)/2 \\
& + \tilde{J}(j)a^+(j+1)a^2(j)/4 + \tilde{J}(j-1)a^+(j-1)a^2(j)/4 \\
& + \tilde{J}(j-1)a^+(j-1)a^2(j-1)/4 + \tilde{J}(j)a^+(j+1)a^2(j+1)/4 \\
& - \tilde{J}(j-1)a^+(j-1)a(j-1)a(j) - \tilde{J}(j)a^+(j+1)a(j+1)a(j) \\
& - \tilde{D}(j)[1 + 2a^+(j)a(j)]a(j)\} \\
& + \varepsilon^6(1/32)\{3\tilde{J}(j-1)a^+(j)a^+(j)a^2(j)a(j-1) \\
& + 3\tilde{J}(j)a^+(j)a^+(j)a^2(j)a(j+1) + 2\tilde{J}(j-1)a^+(j)a(j)a(j-1) \\
& + 2\tilde{J}(j)a^+(j)a(j)a(j+1) - 4\tilde{J}(j-1)a^+(j-1)a^+(j)a^2(j-1)a(j) \\
& - 4\tilde{J}(j)a^+(j+1)a^+(j)a^2(j+1)a(j) \\
& - 2\tilde{J}(j)a^+(j+1)a^+(j+1)a^2(j)a(j+1) \\
& - 2\tilde{J}(j-1)a^+(j-1)a^+(j-1)a^2(j)a(j-1) \\
& + \tilde{J}(j-1)a^+(j-1)a^+(j-1)a^3(j-1) + 2\tilde{J}(j)a^+(j)a^+(j+1)a^3(j) \\
& + \tilde{J}(j-1)a^+(j-1)a^2(j-1) + \tilde{J}(j)a^+(j+1)a^2(j) \\
& + \tilde{J}(j)a^+(j+1)a^+(j+1)a^3(j+1) + 2\tilde{J}(j-1)a^+(j)a^+(j-1)a^3(j) \\
& + \tilde{J}(j)a^+(j+1)a^2(j+1) + \tilde{J}(j-1)a^+(j-1)a^2(j)\} + \mathrm{O}(\varepsilon^8)
\end{aligned} \tag{9}$$

where $\tilde{\omega}_0 = \hbar\omega_0/J_0S_c^2$ and $\tilde{t} = \omega_0 t$ are the dimensionless frequency and time, respectively. ω_0 is the typical frequency of the excitation.

By introducing Glauber's [26] coherent-state representation for Bose operators

$$a(j)|\alpha\rangle = \alpha(j)|\alpha\rangle \tag{10}$$

$$|\alpha\rangle = \prod_j |\alpha(j)\rangle \tag{11}$$

we find that equation (9) is transformed into

$$\begin{aligned}
\mathrm{i}\tilde{\omega}_0\,\partial\alpha(j)/\partial\tilde{t} = {}& \varepsilon^2\{[\tilde{f}(j) + 2\tilde{D}(j) + 2\tilde{J}(j)]\alpha(j) - \tilde{J}(j-1)\alpha(j-1) - \tilde{J}(j)\alpha(j+1)\}\\
&+ \varepsilon^4\{\tilde{J}(j-1)|\alpha(j)|^2\alpha(j-1)/2 + \tilde{J}(j)|\alpha(j)|^2\alpha(j+1)/2\\
&+ \tilde{J}(j)\alpha^*(j+1)\alpha^2(j)/4 + \tilde{J}(j-1)\alpha^*(j-1)\alpha^2(j)/4\\
&+ \tilde{J}(j-1)|\alpha(j-1)|^2\alpha(j-1)/4 + \tilde{J}(j)|\alpha(j+1)|^2\alpha(j+1)/4\\
&- \tilde{J}(j-1)|\alpha(j-1)|^2\alpha(j) - \tilde{J}(j)|\alpha(j+1)|^2\alpha(j)\\
&- \tilde{D}(j)[1 + 2|\alpha(j)|^2\alpha(j)]\alpha(j)\}\\
&+ \varepsilon^6(1/32)\{3\tilde{J}(j-1)|\alpha(j)|^4\alpha(j-1) + 3\tilde{J}(j)|\alpha(j)|^4\alpha(j+1)\\
&+ 2\tilde{J}(j-1)|\alpha(j)|^2\alpha(j-1) + 2\tilde{J}(j)|\alpha(j)|^2\alpha(j+1)\\
&- 4\tilde{J}(j-1)|\alpha(j-1)|^2|\alpha(j)|^2\alpha(j-1)\\
&- 4\tilde{J}(j)|\alpha(j)|^2\alpha(j+1)|^2\alpha(j+1) - 2\tilde{J}(j)|\alpha(j+1)|^2\alpha^*(j+1)\alpha^2(j)\\
&- 2\tilde{J}(j-1)|\alpha(j-1)|^2\alpha^*(j-1)\alpha^2(j) + \tilde{J}(j-1)|\alpha(j-1)|^4\alpha(j-1)\\
&+ 2\tilde{J}(j)|\alpha(j)|^2\alpha^*(j+1)\alpha^2(j) + \tilde{J}(j-1)|\alpha(j-1)|^2\alpha(j-1)\\
&+ \tilde{J}(j)\alpha^2(j)\alpha^*(j+1) + \tilde{J}(j)\,|\alpha(j+1)|^4\alpha(j+1)\\
&+ 2\tilde{J}(j-1)|\alpha(j)|^2\alpha^*(j-1)\alpha^2(j)\\
&+ \tilde{J}(j)|\alpha(j+1)|^2\alpha(j+1) + \tilde{J}(j-1)\alpha^*(j-1)\alpha^2(j)\} + \mathrm{O}(\varepsilon^8)
\end{aligned} \tag{12}$$

where $|\alpha(j)\rangle$ is the coherent-state eigenvector for operator $a(j)$, $\alpha(j)$ is the coherent amplitude and the asterisks denote complex conjugation. If the characteristic wavelength λ_0 of the excitations (in the case of soliton excitation, λ_0 corresponds to the soliton width) is larger than the lattice constant a, we can take the long-wavelength approximation:

$$\sum \rightarrow \frac{1}{\eta}\int \mathrm{d}\tilde{x} \tag{13}$$

$$\begin{aligned}
\alpha(j \pm 1) - \alpha(x \pm a, t) = {}& \alpha \pm \eta\alpha_{\tilde{x}} + (1/2!)\eta^2\alpha_{\tilde{x}\tilde{x}} \pm (1/3!)\eta^3\alpha_{\tilde{x}\tilde{x}\tilde{x}}\\
&+ (1/4!)\eta^4\alpha_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \mathrm{O}(\eta^5)
\end{aligned} \tag{14}$$

and also for $J(j \pm 1)$, $D(j \pm 1)$. Here $\eta = a/\lambda_0$ and $\tilde{x} = x/\lambda_0$. Then equation (12) becomes

$$\begin{aligned}
\mathrm{i}\tilde{\omega}_0\,\partial\alpha/\partial\tilde{t} = {}& \varepsilon^2\{[\tilde{f}(x,t) + 2\tilde{D}(x,t)]\alpha + \eta\tilde{J}_x\alpha - \tfrac{1}{2}\eta^2[(\tilde{J}\alpha)_{\tilde{x}\tilde{x}} + \tilde{J}\alpha_{\tilde{x}\tilde{x}}]\\
&+ \tfrac{1}{6}\eta^3[(\tilde{J}\alpha)_{\tilde{x}\tilde{x}\tilde{x}} - \tilde{J}\alpha_{\tilde{x}\tilde{x}\tilde{x}}] - \tfrac{1}{24}\eta^4[(\tilde{J}\alpha_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \tilde{J}\alpha_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}] + \mathrm{O}(\eta^5)\}\\
&+ \varepsilon^4\{-\tilde{D}(1 + 2|\alpha|^2)\alpha + \eta 0 + \eta^2[\tilde{J}[-\alpha|\alpha_{\tilde{x}}|^2 + \tfrac{1}{2}\alpha^*(\alpha_{\tilde{x}})^2\\
&- \tfrac{1}{2}\alpha^2\alpha^*_{\tilde{x}\tilde{x}}] - \tfrac{1}{2}\tilde{J}_{\tilde{x}}\alpha^2\alpha^*_{\tilde{x}}]
\end{aligned}$$

$$+ \eta^3[\tilde{J}_{\tilde{x}}[\tfrac{1}{4}\alpha^2\alpha^*_{\tilde{x}\tilde{x}} + \tfrac{1}{2}\alpha|\alpha_{\tilde{x}}|^2 - \tfrac{1}{4}\alpha^*(\alpha_{\tilde{x}})^2] + \tfrac{1}{4}\tilde{J}_{\tilde{x}\tilde{x}}\alpha^2\alpha^*_{\tilde{x}}] + O(\eta^4)\} + \varepsilon^6[\tfrac{1}{4}|\alpha|^2\alpha + O(\eta)] + O(\varepsilon^8). \tag{15}$$

All the quantities in equation (15) are dimensionless. ε and η, two small parameters used in the semiclassical approximation and in the long-wavelength approximation, are written explicitly.

3. One-soliton and two-soliton solutions

In the semiclassical approximation and the long-wavelength approximation, ε and η are small quantities. We can assume that

$$\eta = O(\varepsilon) \tag{16}$$

i.e. ε and η have the same order. The physical meaning of this assumption and the other cases of the relations between ε and η will be discussed in the next section. In order to include the lowest-order non-linear effect, we should retain the terms in equation (15) up to $O(\varepsilon^4)$. Then it is reduced to a non-linear Schrödinger-like equation with variable coefficients

$$|\alpha_t = [g\mu_B f(x,t) + 2S_c(1 - 1/2S)D(x,t)]\alpha + aS_cJ_x\alpha - (S_ca^2/2)[(J\alpha)_{xx} + J\alpha_{xx}] - (2S_c/S)D(x,t)|\alpha|^2\alpha \tag{17}$$

when returning to dimensional variables. In the homogeneous case (i.e. $J(x,t) = J_0$, $D(x,t) = D_0$ and $f(x,t) = f_0$), equation (17) takes the form

$$i\alpha_t = [g\mu_B f_0 + 2S_c(1 - 1/2S)D_0]\alpha - J_0a^2S_c\alpha_{xx} - (2S_c/S)D_0|\alpha|^2\alpha. \tag{18}$$

This is the non-linear Schrödinger equation, which is a completely integrable system and can be solved exactly by the inverse scattering transform [27]. A single-soliton solution is

$$\alpha(x,t) = (J_0Sk_0^2a^2/D_0)^{1/2}\operatorname{sech}[k_0(x - x_0 + 2J_0S_ca^2kt] \times \exp[-ikx - i\omega t - i\varphi_0] \tag{19}$$

$$\hbar\omega = g\mu_B f_0S_c/S + 2(1 - 1/2S)S_c^2D_0/S + J_0(S_c^2/S)a^2(k^2 - k_0^2) \tag{20}$$

where x_0, φ_0, k_0 and k are integral constants. Equation (19) represents a wave packet travelling to the left with velocity $2J_0S_ca^2k$. If k is set to zero, it localizes at position $x = x_0$ and oscillates with frequency ω. This type of magnon localization results from the interaction of the magnon via the non-linearity and dispersion of the system. The two-soliton bound-state solution is given by [28]

$$\alpha = (J_0Sa^2/D_0)^{1/2}Q\{k_1\operatorname{sech}[k_1(x + x_0)] \times \exp(-i\omega_1t) + k_2\operatorname{sech}[k_2(x - x_0)]\exp(-i\omega_2t)\} \tag{21}$$

with

$$Q = (k_2^2 - k_1^2)/[\![(k_2^2 + k_1^2) - 2k_1k_2\{\tanh[k_1(x + x_0)]\tanh[k_2(x - x_0)] - \operatorname{sech}[k_1(x + x_0]\operatorname{sech}[k_2(x - x_0)]\cos(\omega t)\}]\!] \tag{22}$$

$$\hbar\omega_1 = g\mu_B f_0 S_c/S + 2S_c^2 D_0(1 - 1/2S)/S - J_0(S_c^2/S)a^2k_1^2 \tag{23}$$

$$\hbar\omega_2 = g\mu_B f_0 S_c/S + 2S_c^2 D_0(1 - 1/2S)/S - J_0(S_c^2/S)a^2k_2^2 \tag{24}$$

$$\hbar\omega = -J_0(S_c^2/S)a^2(k_2^2 - k_1^2) = \hbar(\omega_2 - \omega_1) \tag{25}$$

where k_1, k_2 and x_0 are integral constants. Equation (21) represents two solitons, one vibrating around the equilibrium position $x = -x_0$ with frequency ω_1 and the other around $x = x_0$ with frequency ω_2. The mutual interaction between them is described by the function Q in equation (22). It may be called the two-magnon bound state in the Heisenberg ferromagnetic chain.

In the coherent-state representation, the energy of the single soliton (19) is

$$\begin{aligned} E &= \langle\alpha|(H - H_0)|\alpha\rangle/\langle\alpha|\alpha\rangle = \int \mathrm{d}x\, \mathcal{H}(x, t) \\ &= (2J_0S_ck_0a/D_0)[g\mu_B f_0 + 2S_cD_0(1 - 1/2S) + J_0S_ca^2(k_0^2 + k^2)] \\ &\quad - 8J_0^2S_c^2k_0a^3/3D_0 \end{aligned} \tag{26}$$

where $\mathcal{H}(x, t)$ is the energy density of the system. The spatial configuration of the spins is given by

$$\langle S^z(j)\rangle = \langle\alpha|[S - \alpha^+(j)a(j)]|\alpha\rangle/\langle\alpha|\alpha\rangle. \tag{27}$$

For a single soliton, we have

$$\begin{aligned} \langle S^z(j)\rangle \to \langle S^z(x, t)\rangle &= S\{1 - (J_0k_0^2a^2/D_0) \\ &\quad \times \operatorname{sech}^2[k_0(x - x_0 + 2S_cJ_0a^2kt)]\} \end{aligned} \tag{28}$$

and, for the two-soliton bound state, we find that

$$\begin{aligned} \langle S^z(j)\rangle \to \langle S^z(x, t)\rangle &= S - S(J_0a^2/D_0)Q^2\{k_1^2 \operatorname{sech}^2[k_1(x + x_0)] \\ &\quad + k_2^2 \operatorname{sech}^2[k_2(x - x_0)] + 2k_1k_2 \\ &\quad \times \operatorname{sech}[k_1(x + x_0)] \operatorname{sech}[k_2(x - x_0)] \cos(\omega t)\} \end{aligned} \tag{29}$$

where Q, ω_1, ω_2 and ω are expressed in equations (22)–(25). Equations (28) and (29) show explicitly the localization of the magnon and two-magnon bound state in the Heisenberg ferromagnetic chain.

4. Other relations between η and ε

All the work so far on non-linear excitations in the Heisenberg ferromagnet in the HP transformation has been based on the semiclassical approximation and the long-wavelength approximation [17–21]. The two approximations have been considered to be independent of each other. In the recent work [23, 24], we have shown that the relative ratio of ε to η is very important for determining the modified terms of the equation of motion. For a given physical system, ε and η are related by the characteristic quantities of the system. That is to say $\eta = g(\varepsilon)$ (g is a function of ε). Theoretically, we cannot determine which case is important because different cases correspond to the different physical pictures. Only from the experimental conditions and the initial exciting conditions, can one estimate which case is more suitable. The reason is the same as for the non-linear theory of shallow water wave [29] in which two approximations (small-amplitude approximation and long-wavelength approximation) are also used. In the last section we have discussed only the case $\eta = O(\varepsilon)$. Equation (18) is simply the equation obtained by Pushkarov and Pushkarov [17] but we have generalized it to the inhomogeneous case (equation (17)). If single-ion anisotropy is absent ($D = 0$), equation (17) (and also equation (18)) is invalid for the description of non-linear excitations of the system because the non-linear term vanishes. We must retain the terms to $O(\varepsilon^6)$ in equation (15). Then the equation of motion takes the form

$$\begin{aligned} \mathrm{i}\alpha_t = g\mu_{\mathrm{B}} f(x,t)\alpha + aS_{\mathrm{c}}J_x\alpha - \tfrac{1}{2}a^2S_{\mathrm{c}}[(J\alpha)_{xx} + J\alpha_{xx}] \\ + \tfrac{1}{6}a^3S_{\mathrm{c}}[(J\alpha)_{xxx} - J\alpha_{xxx}] - \tfrac{1}{24}a^4S_{\mathrm{c}}[(J\alpha)_{xxxx} + J\alpha_{xxxx}] \\ + (S_{\mathrm{c}}a^2/S)\{J[-\alpha|\alpha_x|^2 + \tfrac{1}{2}\alpha^*(\alpha_x)^2 - \tfrac{1}{2}\alpha^2\alpha^*_{xx}] - \tfrac{1}{2}J_x\alpha^2\alpha^*_x\} \end{aligned} \tag{30}$$

when returning to dimensional variables.

Next, we discuss other relations between ε and η. The general consideration is that, in order to make the perturbation theory valid for the description of non-linear excitations of the system, the lowest-order non-linear terms should be included in the reduced equation of motion.

4.1. *The case $\eta = O(\varepsilon^{3/2})$*

Retaining the terms to $O(\varepsilon^5)$, we obtain

$$\begin{aligned} 1\alpha_t = [g\mu_{\mathrm{B}} f(x,t) + 2S_{\mathrm{c}}(1 - 1/2S)D(x,t) + aS_{\mathrm{c}}J_x]\alpha \\ - \tfrac{1}{2}a^2S_{\mathrm{c}}[(J\alpha)_{xx} + J\alpha_{xx}] - 2(S_{\mathrm{c}}/S)D(x,t)|\alpha|^2\alpha. \end{aligned} \tag{31}$$

If $D(x,t) = 0$, we must retain the terms to $O(\varepsilon^7)$. Then we have

$$\begin{aligned} \mathrm{i}\alpha_t = [g\mu_{\mathrm{B}} f(x,t) + S_{\mathrm{c}}aJ_x]\alpha - \tfrac{1}{2}a^2S_{\mathrm{c}}[(J\alpha)_{xx} + J\alpha_{xx}] + \tfrac{1}{6}a^3S_{\mathrm{c}}[(J\alpha)_{xxx} - J\alpha_{xxx}] \\ + (S_{\mathrm{c}}/S)a^2\{J[-\alpha|\alpha_x|^2 + \tfrac{1}{2}\alpha^*(\alpha_x)^2 - \tfrac{1}{2}\alpha^2\alpha^*_{xx}] \\ - \tfrac{1}{2}J_x\alpha^2\alpha^*_x\} + (1/4S^2)|\alpha|^2\alpha. \end{aligned} \tag{32}$$

For a homogeneous system, (32) reads

$$\begin{aligned} \mathrm{i}\alpha_t = g\mu_{\mathrm{B}} f_0\alpha - a^2S_{\mathrm{c}}J_0\alpha_{xx} + (1/4S^2)|\alpha|^2\alpha + (S_{\mathrm{c}}/S)a^2J_0[-\alpha|\alpha_x|^2 + \tfrac{1}{2}\alpha^*(\alpha_x)^2 \\ - \tfrac{1}{2}\alpha^2\alpha^*_{xx}]. \end{aligned} \tag{33}$$

This is simply the equation obtained by de Azevedo *et al* [18].

4.2. *The case $\eta = O(\varepsilon^2)$*

Retaining the terms to $O(\varepsilon^6)$, we obtain

$$\begin{aligned} \mathrm{i}\alpha_t = [g\mu_{\mathrm{B}} f(x,t) + 2S_{\mathrm{c}}(1 - 1/2S)D(x,t) + aS_{\mathrm{c}}J_x]\alpha \\ - \tfrac{1}{2}a^2S_{\mathrm{c}}[(J\alpha)_{xx} + J\alpha_{xx}] - 2(S_{\mathrm{c}}/S)D(x,t)|\alpha|^2\alpha. \end{aligned} \tag{34}$$

If $D(x,t) = 0$, the higher-order terms of ε must be included.

4.3. *The case $\eta = O(\varepsilon^{1/2})$*

In order to include the lowest-order non-linear term, we must retain the terms to $O(\varepsilon^4)$

in equation (15). This gives

$$
\begin{aligned}
\mathrm{i}\alpha_t = {} & [g\mu_{\mathrm{B}}f(x,t) + 2S_{\mathrm{c}}(1 - 1/2S)D(x,t) + aS_{\mathrm{c}}J_x]\alpha - \tfrac{1}{2}a^2S_{\mathrm{c}}[(J\alpha)_{xx} + J\alpha_{xx}] \\
& + \tfrac{1}{6}a^3S_{\mathrm{c}}[(J\alpha)_{xxx} - J\alpha_{xxx}] - \tfrac{1}{24}a^4S_{\mathrm{c}}[(J\alpha)_{xxxx} + J\alpha_{xxxx}] \\
& - 2(S_{\mathrm{c}}/S)D(x,t)|\alpha|^2\alpha \qquad (35)
\end{aligned}
$$

involving the second-, third- and fourth-order derivatives of α. It is a non-linear Schrödinger equation with variable coefficients and higher-order dispersion effect. Physically, this is the more accurate equation for the description of the non-linearity, inhomogeneity and discreteness of the system.

5. Discussion and summary

It is well known that there are several boson representations of spin operators. Besides the HP transformation, the Dyson–Maleev [30, 31] representation introduces one set of boson operators to replace spin operators. When substituting this transformation into (1), one can obtain an exact Hamiltonian in which there is no term higher than fourth order. Because the Hamiltonian becomes non-Hermite, this procedure will introduce some unphysical states. In the Schwinger [32] boson representation, two sets of boson operators are introduced. With use of this, Cieplak and Turski [22] investigated soliton excitations in a homogeneous ferromagnetic chain in the continuum limit. They obtained an effective Hamiltonian after some quartic terms of boson operators were neglected (this results in only one set of bosons being retained). Instead of the boson representation, Balakrishnan and Bishop [16], using the spin-coherent state, studied non-linear excitations in an isotropic quantum ferromagnetic chain in the continuum approximation (the Hamiltonian is expanded to a^2 order, where a is a lattice constant). This is a useful method for the study of non-linear dynamics in a magnetic chain but it seems that there are some difficulties in the anisotropic case.

The HP representation has been widely used in spin-wave theory. We think that this is because the approach has many advantages. When using this representation to investigate the non-linear excitations in spin system, attention should be paid to the relative ratio of ε to η for determining the modified terms of the equation of motion. In fact, in any perturbation theory involving two or more expansions, one should write the equation system in dimensionless form, compare the relative order of the parameters appearing in the system and then estimate which case is suitable for given physical conditions.

The inhomogeneous spin system can occur in layered materials which can exhibit a quasi-one-dimensional character and other materials may well be grown synthetically in a layered manner by molecular beam epitaxy. In this case the equation of motion such as (17) has periodic coefficients which may admit the so-called gap soliton solution. The concept of the gap soliton was first introduced by Chen and Mills [33] when they studied the non-linear optical response of a superlattice. The approach given above is the first step in studying the gap soliton in the Heisenberg ferromagnetic chain.

In summary, we have studied the magnon localization and two-magnon bound state in the Heisenberg ferromagnetic chain. With use of the HP transformation and Glauber's coherent-state representation, the equation of motion for the operator $a(j)$ is reduced to a non-linear Schrödinger-like equation in the semiclassical approximation and in the long-wave approximation. For the homogeneous system, the exact and explicit single-soliton (localized magnon state) and two-soliton bound-state (two-magnon bound-state) solutions are given by the inverse scattering transform. The results show that magnon localization and a two-magnon bound state are possible in the Heisenberg ferromagnetic chain. The importance of the relative ratio of ε to η is emphasized. Several relations between ε and η are also discussed.

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